Overview

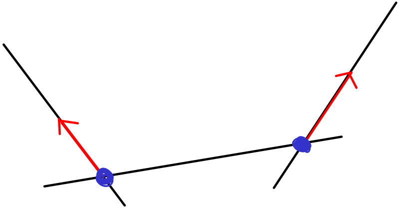
Minkowski (1896) and Weyl (1935)

Outer and Inner Representation of a Polyhedron

For $P \subseteq \mathbb{R}^n$, the following are equivalent:

1. P is a polyhedron
2. There are finite sets $Q, R \subseteq \mathbb{R}^n$ such that $P = \text{conv}(Q) + \text{cone}(R)$
   // P is finitely generated

- choose Q (resp. R) as extreme points (resp. rays) of P



Dantzig-Wolfe Reformulation for LPs (1960, 1961)

- we use this to equivalently reformulate what we call the

original model

$$\begin{array}{rrcl} \min & \mathbf{c}^t\mathbf{x} & & \\ \text{s.t.} & A\mathbf{x} & \geq & \mathbf{b} \\ & D\mathbf{x} & \geq & \mathbf{d} \\ & \mathbf{x} & \geq & \mathbf{0} \end{array}$$

Dantzig-Wolfe Reformulation for LPs (1960, 1961)

- we use this to equivalently reformulate what we call the

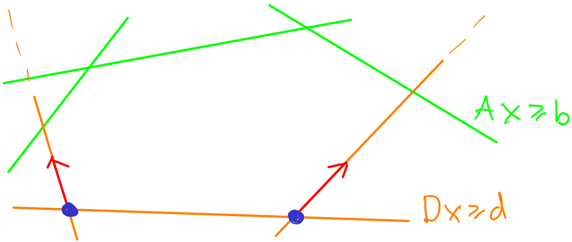
original model
$$\begin{array}{rrcl} \min & \mathbf{c}^t\mathbf{x} & & \\ \text{s.t.} & A\mathbf{x} & \geq & \mathbf{b} \\ & D\mathbf{x} & \geq & \mathbf{d} \\ & \mathbf{x} & \geq & \mathbf{0} \end{array}$$

- identify two sets of constraints, typically constraints we know how to deal (well) with (the "easy constraints") and everything else (the "complicating constraints").

Dantzig-Wolfe Reformulation for LPs (1960, 1961)

original formulation
$$\begin{aligned} z^*_{\text{LP}} = \min \quad & \mathbf{c}^t\mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} \geq \mathbf{b} \\ & D\mathbf{x} \geq \mathbf{d} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Idea: apply Minkowski-Weyl on the "easy constraints" $X = \{\mathbf{x} \geq \mathbf{0} \mid D\mathbf{x} \geq \mathbf{d}\}$



vertices $Q = \{\mathbf{x}_1, \dots, \mathbf{x}_{|Q|}\}$, extreme rays $R = \{\mathbf{x}_1, \dots, \mathbf{x}_{|R|}\}$ of X

Operations Research RWTH AACHEN UNIVERSITY

Dantzig-Wolfe Reformulation for LPs (1960, 1961)

vertices $Q = \{\mathbf{x}_1, \ldots, \mathbf{x}_{|Q|}\}$, extreme rays $R = \{\mathbf{x}_1, \ldots, \mathbf{x}_{|R|}\}$ of X

express every $\mathbf{x} \in X$ as

$$\begin{aligned}
\mathbf{x} = \sum_{q \in Q} \lambda_q \mathbf{x}_q \quad + \quad \sum_{r \in R} \lambda_r \mathbf{x}_r & & & \\
\sum_{q \in Q} \lambda_q \qquad\qquad\qquad & = 1 & & \text{// convexity constraint} \\
\lambda_q \qquad\qquad\qquad & \geq 0 & & q \in Q \\
\lambda_r & \geq 0 & & r \in R
\end{aligned}$$

and substitute this $\mathbf{x} \in X$ in $A\mathbf{x} \geq \mathbf{b}$ and $\mathbf{c}^t\mathbf{x}$.

Dantzig-Wolfe Reformulation for LPs (1960, 1961)

substitution of $\mathbf{x} \in X$ in $A\mathbf{x} \geq \mathbf{b}$ and $\mathbf{c}^t\mathbf{x}$

$$
\begin{array}{lllll}
\min & \mathbf{c}^t \left(\sum_{q \in Q} \lambda_q \mathbf{x}_q + \sum_{r \in R} \lambda_r \mathbf{x}_r \right) & & & \\
\text{s.t.} & A \left(\sum_{q \in Q} \lambda_q \mathbf{x}_q + \sum_{r \in R} \lambda_r \mathbf{x}_r \right) & \geq \mathbf{b} & & \\
& \sum_{q \in Q} \lambda_q & = 1 & & \\
& \lambda_q & \geq 0 & & q \in Q \\
& \lambda_r & \geq 0 & & r \in R
\end{array}
$$

Dantzig-Wolfe Reformulation for LPs (1960, 1961)

substitution of $\mathbf{x} \in X$ in $A\mathbf{x} \geq \mathbf{b}$ and $\mathbf{c}^t\mathbf{x}$ and some rearranging

$$
\begin{array}{llllll}
\min & \sum_{q \in Q} \lambda_q \mathbf{c}^t \mathbf{x}_q & + & \sum_{r \in R} \lambda_r \mathbf{c}^t \mathbf{x}_r & & \\
\text{s.t.} & \sum_{q \in Q} \lambda_q A \mathbf{x}_q & + & \sum_{r \in R} \lambda_r A \mathbf{x}_r & \geq \mathbf{b} & \\
& \sum_{q \in Q} \lambda_q & & & = 1 & \\
& \lambda_q & & & \geq 0 & q \in Q \\
& & & \lambda_r & \geq 0 & r \in R
\end{array}
$$

Dantzig-Wolfe Reformulation for LPs (1960, 1961)

substitution of $\mathbf{x} \in X$ in $A\mathbf{x} \geq \mathbf{b}$ and $\mathbf{c}^t\mathbf{x}$ and some rearranging

$$
\begin{aligned}
\min \quad & \sum_{q \in Q} \lambda_q \underbrace{\mathbf{c}^t \mathbf{x}_q}_{=: c_q} + \sum_{r \in R} \lambda_r \underbrace{\mathbf{c}^t \mathbf{x}_r}_{=: c_r} \\
\text{s.t.} \quad & \sum_{q \in Q} \lambda_q \underbrace{A \mathbf{x}_q}_{=: \mathbf{a}_q} + \sum_{r \in R} \lambda_r \underbrace{A \mathbf{x}_r}_{=: \mathbf{a}_r} \geq \mathbf{b} \\
& \sum_{q \in Q} \lambda_q = 1 \\
& \lambda_q \geq 0 \qquad q \in Q \\
& \lambda_r \geq 0 \qquad r \in R
\end{aligned}
$$

Dantzig-Wolfe Reformulation for LPs (1960, 1961)

leads to an *extended* LP which we call the *master problem*

$$
\begin{aligned}
z^*_{\mathrm{MP}} \;=\; \min \quad & \sum_{q\in Q} c_q\lambda_q + \sum_{r\in R} c_r\lambda_r \\
\text{s.t.} \quad & \sum_{q\in Q} \mathbf{a}_q\lambda_q + \sum_{r\in R} \mathbf{a}_r\lambda_r \geq \mathbf{b} \\
& \sum_{q\in Q} \lambda_q = 1 \\
& \lambda_q \geq 0 \qquad q\in Q \\
& \lambda_r \geq 0 \qquad r\in R
\end{aligned}
$$

which is *equivalent* to the original LP, i.e., $z^*_{\mathrm{LP}} = z^*_{\mathrm{MP}}$

The Dantzig-Wolfe Master Problem

- the master problem has a huge number $|Q| + |R|$ of variables
- it needs to be solved by column generation
- initialize the RMP with $Q' \subseteq Q$ and $R' \subseteq R$
- solve the RMP to obtain primal $\boldsymbol{\lambda}$ and dual $\boldsymbol{\pi}, \pi_0$

The Dantzig-Wolfe Restricted Master Problem

$$
\begin{aligned}
z^*_{\mathrm{RMP}} \;=\; \min \quad & \sum_{q \in Q'} c_q \lambda_q + \sum_{r \in R'} c_r \lambda_r \\
\text{s.t.} \quad & \sum_{q \in Q'} \mathbf{a}_q \lambda_q + \sum_{r \in R'} \mathbf{a}_r \lambda_r \geq \mathbf{b} && [\boldsymbol{\pi}] \\
& \sum_{q \in Q'} \lambda_q = 1 && [\pi_0] \\
& \lambda_q \geq 0 && q \in Q' \\
& \lambda_r \geq 0 && r \in R'
\end{aligned}
$$

Reduced Cost Computation

- for the reduced cost formula we distinguish two cases

→ for λ_q, $q \in Q$: // variables corresponding to extreme points

$$\bar{c}_q = c_q - (\boldsymbol{\pi}^t, \pi_0) \begin{pmatrix} \mathbf{a}_q \\ 1 \end{pmatrix} = c_q - \boldsymbol{\pi}^t \mathbf{a}_q - \pi_0$$

$$= \mathbf{c}^t \mathbf{x}_q - \boldsymbol{\pi}^t A \mathbf{x}_q - \pi_0$$

→ for λ_r, $r \in R$: // variables corresponding to extreme rays

$$\bar{c}_r = c_r - (\boldsymbol{\pi}^t, \pi_0) \begin{pmatrix} \mathbf{a}_r \\ 0 \end{pmatrix} = c_r - \boldsymbol{\pi}^t \mathbf{a}_r$$

$$= \mathbf{c}^t \mathbf{x}_r - \boldsymbol{\pi}^t A \mathbf{x}_r$$

- we need to compute $\bar{c}^* = \min\{\min_{q \in Q} \bar{c}_q, \min_{r \in R} \bar{c}_r\}$ // the smallest reduced cost

Dantzig-Wolfe Pricing Problem

- in words: find an extreme point $q \in Q$ with minimum $\bar{c}_q$ and/or an extreme ray $r \in R$ with minimum $\bar{c}_r$

Dantzig-Wolfe Pricing Problem

- in words: find an extreme point $q \in Q$ with minimum $\bar{c}_q$ and/or an extreme ray $r \in R$ with minimum $\bar{c}_r$
- to this end, solve the *Dantzig-Wolfe pricing problem*

$$z^*_{\mathrm{PP}} = \min_{j \in Q \cup R} \quad \mathbf{c}^t \mathbf{x}_j - \boldsymbol{\pi}^t A \mathbf{x}_j \qquad \text{// no } \pi_0 \text{ here}$$

Dantzig-Wolfe Pricing Problem

- in words: find an extreme point $q \in Q$ with minimum $\bar{c}_q$ and/or an extreme ray $r \in R$ with minimum $\bar{c}_r$
- to this end, solve the *Dantzig-Wolfe pricing problem*

$$
\begin{array}{rcl}
z^\star_{\mathrm{PP}} & = & \min\limits_{j \in Q \cup R} \quad \mathbf{c}^t \mathbf{x}_j - \boldsymbol{\pi}^t A \mathbf{x}_j \qquad \text{// no } \pi_0 \text{ here} \\
 & = & \begin{array}{rrcl} \min & (\mathbf{c}^t - \boldsymbol{\pi}^t A)\mathbf{x} & & \\ \text{s.t.} & D\mathbf{x} & \geq & \mathbf{d} \\ & \mathbf{x} & \geq & \mathbf{0} \end{array}
\end{array}
$$

- Q and R contain the extreme points/extreme rays of $\{\mathbf{x} \geq \mathbf{0} \mid D\mathbf{x} \geq \mathbf{d}\}$!
- the pricing problem is again a linear program // solve it e.g., with the simplex algorithm

Dantzig-Wolfe Pricing Problem

three cases for $z^*_{\mathrm{PP}} = \min_{\mathbf{x} \geq \mathbf{0}} \left\{ (\mathbf{c}^t - \boldsymbol{\pi}^t A)\mathbf{x} \mid D\mathbf{x} \geq \mathbf{d} \right\}$

Dantzig-Wolfe Pricing Problem

three cases for $z^*_{\mathrm{PP}} = \min_{\mathbf{x} \geq \mathbf{0}} \left\{ (\mathbf{c}^t - \boldsymbol{\pi}^t A)\mathbf{x} \mid D\mathbf{x} \geq \mathbf{d} \right\}$

1. $z^*_{\mathrm{PP}} = -\infty$

Dantzig-Wolfe Pricing Problem

three cases for $z^*_{\text{PP}} = \min_{\mathbf{x} \geq \mathbf{0}} \{(\mathbf{c}^t - \boldsymbol{\pi}^t A)\mathbf{x} \mid D\mathbf{x} \geq \mathbf{d}\}$

1. $z^*_{\text{PP}} = -\infty \Rightarrow$ we identified an extreme ray $r^* \in R$ with $\bar{c}_{r^*} < 0$
   $\rightarrow$ add variable λ_{r^*} to the RMP
   with cost $\mathbf{c}^t \mathbf{x}_{r^*}$ and column coefficients $\begin{pmatrix} A\mathbf{x}_{r^*} \\ 0 \end{pmatrix}$

Dantzig-Wolfe Pricing Problem

three cases for $z^\star_{\text{PP}} = \min_{\mathbf{x}\geq\mathbf{0}} \{(\mathbf{c}^t - \boldsymbol{\pi}^t A)\mathbf{x} \mid D\mathbf{x} \geq \mathbf{d}\}$

1. $z^\star_{\text{PP}} = -\infty \Rightarrow$ we identified an extreme ray $r^\ast \in R$ with $\bar{c}_{r^\ast} < 0$
   $\rightarrow$ add variable $\lambda_{r^\ast}$ to the RMP
   with cost $\mathbf{c}^t\mathbf{x}_{r^\ast}$ and column coefficients $\begin{pmatrix} A\mathbf{x}_{r^\ast} \\ 0 \end{pmatrix}$

2. $-\infty < z^\star_{\text{PP}} - \pi_0 < 0$

Dantzig-Wolfe Pricing Problem

three cases for $z^*_{\mathrm{PP}} = \min_{\mathbf{x}\geq\mathbf{0}} \{(\mathbf{c}^t - \boldsymbol{\pi}^t A)\mathbf{x} \mid D\mathbf{x} \geq \mathbf{d}\}$

1. $z^*_{\mathrm{PP}} = -\infty \Rightarrow$ we identified an extreme ray $r^* \in R$ with $\bar{c}_{r^*} < 0$
   $\rightarrow$ add variable λ_{r^*} to the RMP
   with cost $\mathbf{c}^t\mathbf{x}_{r^*}$ and column coefficients $\begin{pmatrix} A\mathbf{x}_{r^*} \\ 0 \end{pmatrix}$

2. $-\infty < z^*_{\mathrm{PP}} - \pi_0 < 0 \Rightarrow$ we identified an extreme point $q^* \in Q$ with $\bar{c}_{q^*} < 0$
   $\rightarrow$ add variable λ_{q^*} to the RMP
   with cost $\mathbf{c}^t\mathbf{x}_{q^*}$ and column coefficients $\begin{pmatrix} A\mathbf{x}_{q^*} \\ 1 \end{pmatrix}$

Dantzig-Wolfe Pricing Problem

three cases for $z^\star_{\text{PP}} = \min_{\mathbf{x}\geq\mathbf{0}} \{(\mathbf{c}^t - \boldsymbol{\pi}^t A)\mathbf{x} \mid D\mathbf{x} \geq \mathbf{d}\}$

1. $z^\star_{\text{PP}} = -\infty \Rightarrow$ we identified an extreme ray $r^* \in R$ with $\bar{c}_{r^*} < 0$
   $\rightarrow$ add variable λ_{r^*} to the RMP
   with cost $\mathbf{c}^t\mathbf{x}_{r^*}$ and column coefficients $\begin{pmatrix} A\mathbf{x}_{r^*} \\ 0 \end{pmatrix}$
2. $-\infty < z^\star_{\text{PP}} - \pi_0 < 0 \Rightarrow$ we identified an extreme point $q^* \in Q$ with $\bar{c}_{q^*} < 0$
   $\rightarrow$ add variable λ_{q^*} to the RMP
   with cost $\mathbf{c}^t\mathbf{x}_{q^*}$ and column coefficients $\begin{pmatrix} A\mathbf{x}_{q^*} \\ 1 \end{pmatrix}$
3. $0 \leq z^\star_{\text{PP}} - \pi_0$

Dantzig-Wolfe Pricing Problem

three cases for $z^\star_{\text{PP}} = \min_{\mathbf{x}\geq\mathbf{0}} \{(\mathbf{c}^t - \boldsymbol{\pi}^t A)\mathbf{x} \mid D\mathbf{x} \geq \mathbf{d}\}$

1. $z^\star_{\text{PP}} = -\infty \Rightarrow$ we identified an extreme ray $r^\star \in R$ with $\bar{c}_{r^\star} < 0$
   $\rightarrow$ add variable $\lambda_{r^\star}$ to the RMP
   with cost $\mathbf{c}^t\mathbf{x}_{r^\star}$ and column coefficients $\begin{pmatrix} A\mathbf{x}_{r^\star} \\ 0 \end{pmatrix}$
2. $-\infty < z^\star_{\text{PP}} - \pi_0 < 0 \Rightarrow$ we identified an extreme point $q^\star \in Q$ with $\bar{c}_{q^\star} < 0$
   $\rightarrow$ add variable $\lambda_{q^\star}$ to the RMP
   with cost $\mathbf{c}^t\mathbf{x}_{q^\star}$ and column coefficients $\begin{pmatrix} A\mathbf{x}_{q^\star} \\ 1 \end{pmatrix}$
3. $0 \leq z^\star_{\text{PP}} - \pi_0 \Rightarrow$ *there is no* $j \in Q \cup R$ with $\bar{c}_j < 0$.

Projecting back to the Original Variables

- by construction, we can always obtain an original $\mathbf{x}$ solution from a master $\boldsymbol{\lambda}$ solution via

$$\mathbf{x} = \sum_{q \in Q} \lambda_q \mathbf{x}_q + \sum_{r \in R} \lambda_r \mathbf{x}_r$$

Briefly Pause

- go back to the cutting stock and vertex coloring problems

- you recognize original constraints in master and pricing problems
- however, not exactly, the reformulation “forgot” about rolls and colors

$\rightarrow$ this is common and called *aggregation*

Block-Angular Matrices

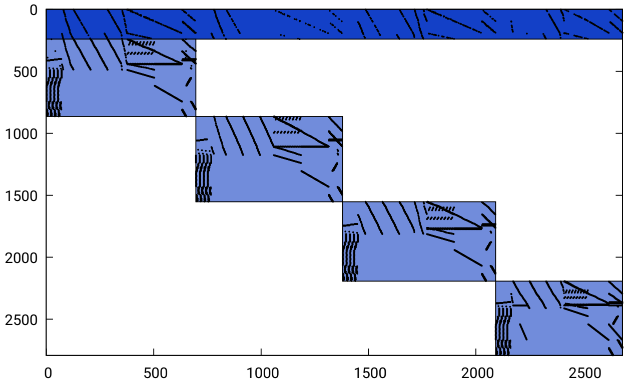
- the classical Dantzig-Wolfe situation is

$$
\begin{array}{llllllll}
\min & \mathbf{c}_1^t\mathbf{x}^1 & + & \mathbf{c}_2^t\mathbf{x}^2 & + \cdots + & \mathbf{c}_K^t\mathbf{x}^K & & \\
\text{s.t.} & A_1\mathbf{x}^1 & + & A_2\mathbf{x}^2 & + \cdots + & A_K\mathbf{x}^K & \geq & \mathbf{b} \\
 & D_1\mathbf{x}^1 & & & & & \geq & \mathbf{d}_1 \\
 & & & D_2\mathbf{x}^2 & & & \geq & \mathbf{d}_2 \\
 & & & & \ddots & & & \vdots \\
 & & & & & D_K\mathbf{x}^K & \geq & \mathbf{d}_K \\
 & \mathbf{x}^1 & , & \mathbf{x}^2 & , \ldots , & \mathbf{x}^K & \geq & \mathbf{0}
\end{array}
$$

- K rolls, K colors, K vehicles, K *subproblems*, ...

Block-Angular Matrices

- the classical Dantzig-Wolfe situation is



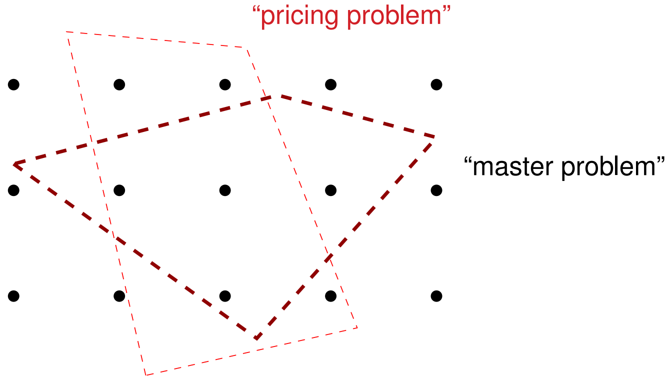
Block-Angular Matrices

- constraints/variables of each *block* are separately DW reformulated
- yields K pricing problems
- *all* must report non-negative reduced cost for RMP optimality

- the blocks can be identical, in which case one can *aggregate* them
  // loosely speaking, master and pricing problem use only one representative

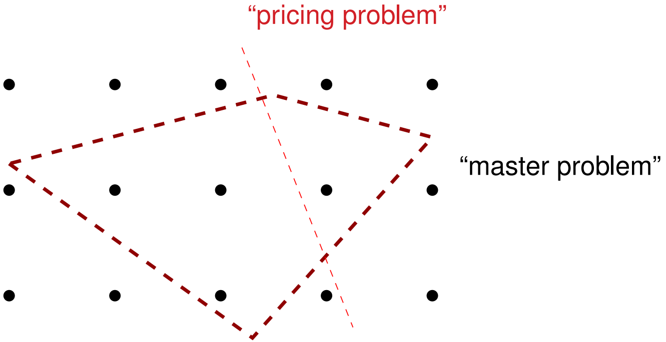
Dantzig-Wolfe Reformulation for LPs: Pictorially

$$\{x \in \mathbb{Q}^n \mid Dx \geq d\} \;\cap\; \{x \in \mathbb{Q}^n \mid Ax \geq b\}$$



Dantzig-Wolfe Reformulation for LPs: Pictorially

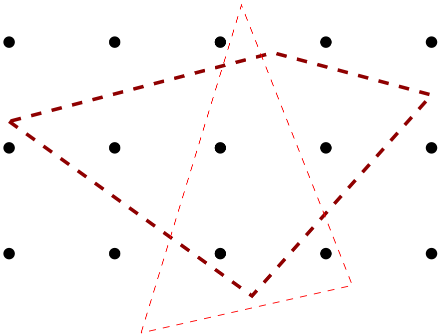
$$\{x \in \mathbb{Q}^n \mid Dx \geq d\} \quad \cap \quad \{x \in \mathbb{Q}^n \mid Ax \geq b\}$$



Dantzig-Wolfe Reformulation for LPs: Pictorially

$$\{x \in \mathbb{Q}^n \mid Dx \geq d\} \;\cap\; \{x \in \mathbb{Q}^n \mid Ax \geq b\}$$

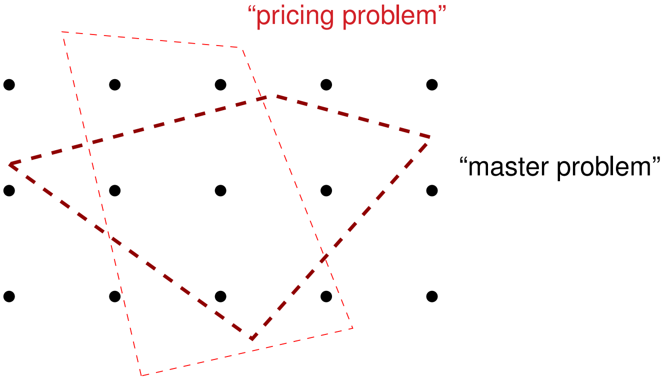
“pricing problem”



“master problem”

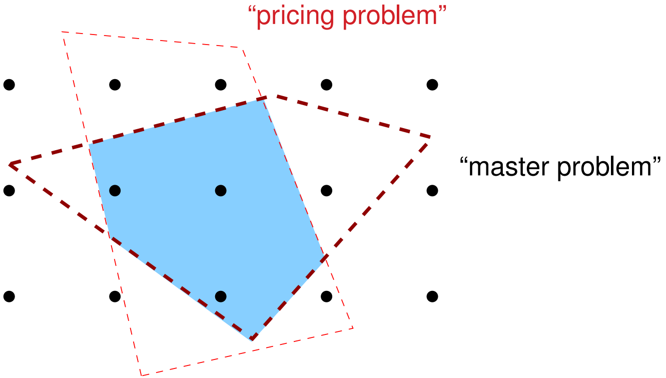
Dantzig-Wolfe Reformulation for LPs: Pictorially

$$\{x \in \mathbb{Q}^n \mid Dx \geq d\} \;\cap\; \{x \in \mathbb{Q}^n \mid Ax \geq b\}$$



Dantzig-Wolfe Reformulation for LPs: Pictorially

$$\{x \in \mathbb{Q}^n \mid Dx \geq d\} \;\cap\; \{x \in \mathbb{Q}^n \mid Ax \geq b\}$$



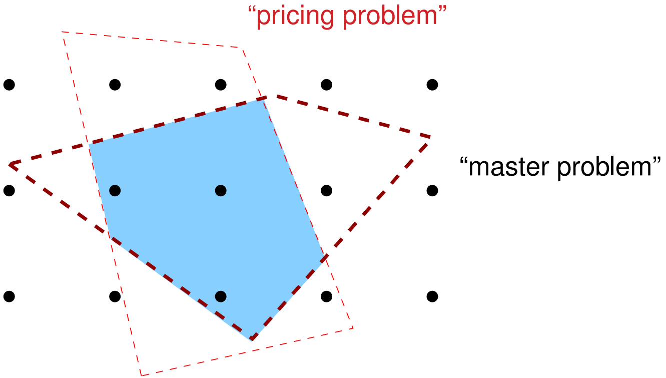
- not tighter than standard LP relaxation

Briefly Pause

- but the pricing problems we have seen were *integer* programs!

Dantzig-Wolfe Reformulation for IPs: Pictorially

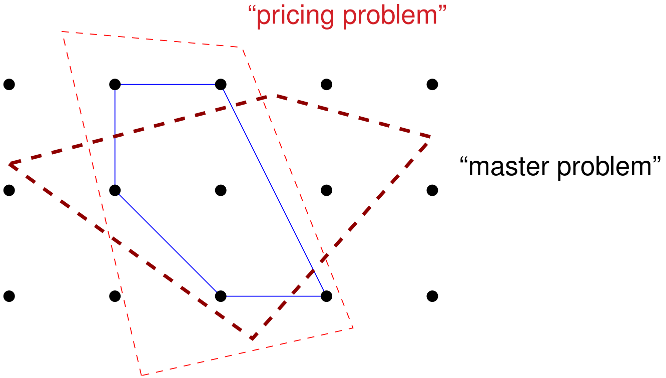
$$\{x \in \mathbb{Q}^n \mid Dx \geq d\} \;\cap\; \{x \in \mathbb{Q}^n \mid Ax \geq b\}$$



- not tighter than standard LP relaxation

Dantzig-Wolfe Reformulation for IPs: Pictorially

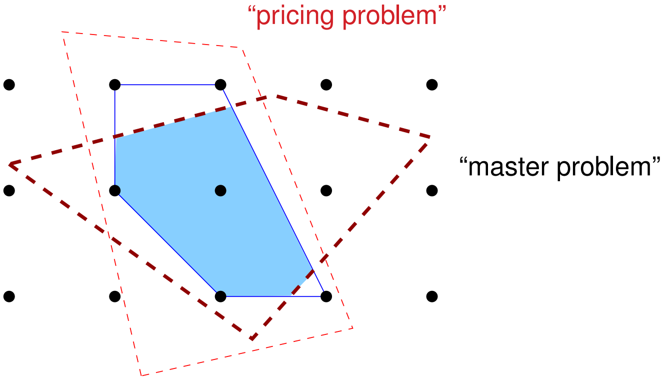
$\{x \in \mathbb{Q}^n \mid Dx \geq d\} \;\cap\; \{x \in \mathbb{Q}^n \mid Ax \geq b\}$



- for integer programs: partial convexification $\mathrm{conv}\{x \in \mathbb{Z}^n \mid Dx \geq d\}$

Dantzig-Wolfe Reformulation for IPs: Pictorially

$$\{x \in \mathbb{Q}^n \mid Dx \geq d\} \cap \{x \in \mathbb{Q}^n \mid Ax \geq b\}$$



- for integer programs: partial convexification, possibly stronger

Do you know it?

- DW reformulating a linear program leads to an equivalent linear program

$\rightarrow$ true

$\rightarrow$ false

$\rightarrow$ it depends

- DW reformulating an integer program leads to a stronger relaxation than the LP relaxation

$\rightarrow$ true

$\rightarrow$ false

$\rightarrow$ it depends

Briefly Pause

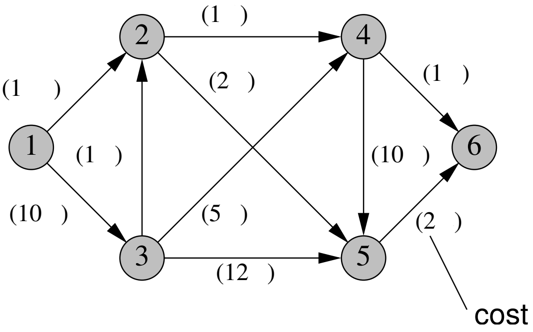
- column generation relies on our ability to optimize over $\operatorname{conv}\{x \in \mathbb{Z}^n \mid Dx \geq d\}$
- how should we choose $D\mathbf{x} \geq \mathbf{d}$?

$\rightarrow$ $D\mathbf{x} \geq \mathbf{d}$ should describe a structure over which we can (easily) optimize

$\rightarrow$ convexifying $D\mathbf{x} \geq \mathbf{d}$ should improve the dual bound (well)

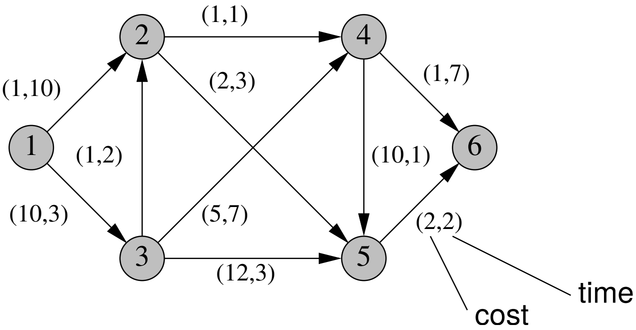
Numerical Example: Taken from the "Primer"

- find: shortest path (RCSP) from 1 to 6



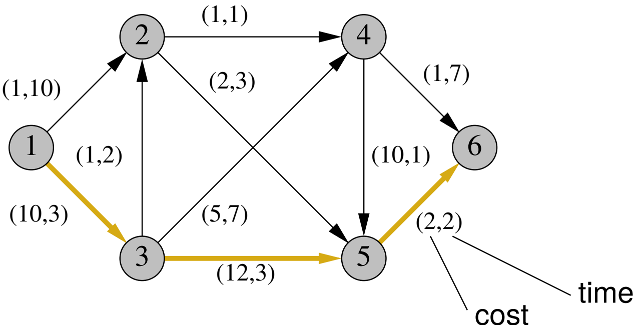
Numerical Example: Taken from the "Primer"

- find: resource constrained shortest path (RCSP) from 1 to 6
- total traversal time must not exceed 14 units



Numerical Example: Taken from the "Primer"

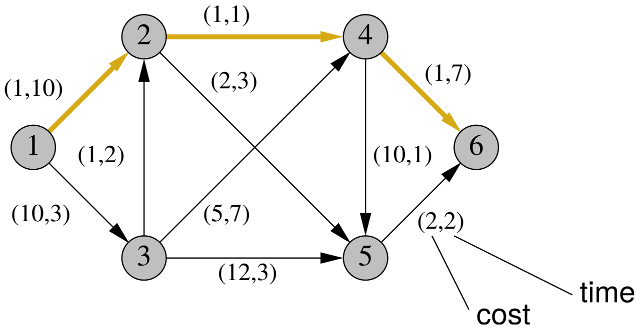
- find: resource constrained shortest path (RCSP) from 1 to 6
- total traversal time must not exceed 14 units



- path 1-3-5-6 is quick but expensive: cost 24, time 8

Numerical Example: Taken from the "Primer"

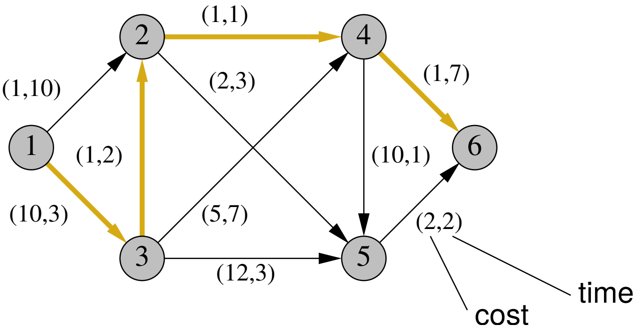
- find: resource constrained shortest path (RCSP) from 1 to 6
- total traversal time must not exceed 14 units



- path 1-2-4-6 is cheap but too slow: cost 3, time 18

Numerical Example: Taken from the “Primer”

- find: resource constrained shortest path (RCSP) from 1 to 6
- total traversal time must not exceed 14 units



- path 1-3-2-4-6 is optimal: cost 13, time 13

Integer Program for the RCSP Problem

- c_{ij} cost on arc (i, j), t_{ij} time to traverse (i, j)

$$
\begin{aligned}
z^\star := \min \sum_{(i,j)\in A} c_{ij} x_{ij} & \\
\text{s.t.} \sum_{j:(1,j)\in A} x_{1j} &= 1 \\
\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} &= 0 \qquad i = 2, 3, 4, 5 \\
\sum_{i:(i,6)\in A} x_{i6} &= 1 \\
\sum_{(i,j)\in A} t_{ij} x_{ij} &\leq 14 \\
x_{ij} &\in \{0, 1\} \quad (i, j) \in A
\end{aligned}
$$

Integer Program for the RCSP Problem

- c_{ij} cost on arc (i,j), t_{ij} time to traverse (i,j)

$$
\begin{aligned}
z^\star := \min \sum_{(i,j)\in A} c_{ij}x_{ij} & \\
\text{s.t.} \sum_{j:(1,j)\in A} x_{1j} &= 1 \\
\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} &= 0 \qquad i = 2,3,4,5 \\
\sum_{i:(i,6)\in A} x_{i6} &= 1 \\
\sum_{(i,j)\in A} t_{ij}x_{ij} &\leq 14 \\
x_{ij} &\in \{0,1\} \qquad (i,j)\in A
\end{aligned}
$$

- could be solved by branch-and-bound (B&B)

Integer Program for the RCSP Problem

- c_{ij} cost on arc (i,j), t_{ij} time to traverse (i,j)

$$
\begin{aligned}
z^\star := \min \sum_{(i,j)\in A} c_{ij}x_{ij} & \\
\text{s.t.} \sum_{j:(1,j)\in A} x_{1j} &= 1 \\
\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} &= 0 \qquad i = 2,3,4,5 \\
\sum_{i:(i,6)\in A} x_{i6} &= 1 \\
\cancel{\sum_{(i,j)\in A} t_{ij}x_{ij} \le 14} & \\
x_{ij} \in \{0,1\} & \quad (i,j) \in A
\end{aligned}
$$

- instead: exploit embedded shortest path problem structure

Paths vs. Arcs Formulation

- what remains // these constraints go into the pricing problem

$$
\begin{aligned}
\sum_{j:(1,j)\in A} x_{1j} &= 1 \\
\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} &= 0 \qquad i = 2,3,4,5 \\
\sum_{i:(i,6)\in A} x_{i6} &= 1 \\
x_{ij} &\in \{0,1\} \quad (i,j) \in A
\end{aligned}
$$

defines a (particular) network flow problem

- fact: every flow defined on arcs decomposes into flows on paths (and cycles)

Paths vs. Arcs Formulation

- the convex hull of

$$\begin{aligned}
\sum_{j:(1,j)\in A} x_{1j} &= 1 \\
\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} &= 0 \qquad i = 2,3,4,5 \\
\sum_{i:(i,6)\in A} x_{i6} &= 1 \\
x_{ij} &\in \{0,1\} \quad (i,j) \in A
\end{aligned}$$

  defines a polyhedron (in fact, a polytope) with integer vertices

- fact: every arc flow can be represented as convex combination of path (and cycle) flows
  // vertices of the above polytope are incidence vectors of 1-6-paths

Paths vs. Arcs Formulation

- fact: every arc flow can be represented as convex combination of path (and cycle) flows

$$
\begin{aligned}
x_{ij} = \sum_{p \in P} x_{pij} \lambda_p & \quad (i,j) \in A \\
\sum_{p \in P} \lambda_p = 1 & \quad \text{// convexity constraint} \\
\lambda_p \geq 0 & \quad p \in P
\end{aligned}
$$

- P denotes the set of *all* paths from node 1 to node 6
- notation: $x_{pij} = 1$ iff arc (i,j) on path p, otherwise $x_{pij} = 0$

Integer Master Problem

- now substitute for x_{ij} in our original IP

$$
\begin{aligned}
\sum_{p \in P} \lambda_p &= 1 && \\
\lambda_p &\geq 0 && p \in P \\
\sum_{p \in P} x_{pij} \lambda_p &= x_{ij} && (i,j) \in A
\end{aligned}
$$

Integer Master Problem

- now substitute for x_{ij} in our original IP

$$
\begin{aligned}
z^\star = \min \sum_{p \in P} \Big(\sum_{(i,j) \in A} c_{ij} x_{pij} \Big) \lambda_p & \\
\text{s.t.} \sum_{p \in P} \Big(\sum_{(i,j) \in A} t_{ij} x_{pij} \Big) \lambda_p \leq 14 & \\
\sum_{p \in P} \lambda_p = 1 & \\
\lambda_p \geq 0 & \qquad p \in P \\
\sum_{p \in P} x_{pij} \lambda_p = x_{ij} & \qquad (i,j) \in A \\
x_{ij} \in \{0,1\} & \qquad (i,j) \in A
\end{aligned}
$$

Master Problem

- now substitute for x_{ij} in our original IP and relax the integrality of x_{ij}

$$
\begin{aligned}
z^\star = \min \sum_{p\in P} \Big(\sum_{(i,j)\in A} c_{ij} x_{pij} \Big) \lambda_p & \\
\text{s.t.} \sum_{p\in P} \Big(\sum_{(i,j)\in A} t_{ij} x_{pij} \Big) \lambda_p &\leq 14 \\
\sum_{p\in P} \lambda_p &= 1 \\
\lambda_p &\geq 0 & p \in P \\
\sum_{p\in P} x_{pij} \lambda_p &= x_{ij} & (i,j) \in A \\
x_{ij} &\geq 0 & (i,j) \in A
\end{aligned}
$$

Master Problem

- now substitute for x_{ij} in our original IP and relax the integrality of x_{ij}

$$
\begin{aligned}
\bar{z}^\star = \min \sum_{p \in P} \Big(\sum_{(i,j) \in A} c_{ij} x_{pij} \Big) \lambda_p & \\
\text{s.t.} \sum_{p \in P} \Big(\sum_{(i,j) \in A} t_{ij} x_{pij} \Big) \lambda_p &\le 14 \\
\sum_{p \in P} \lambda_p &= 1 \\
\lambda_p &\ge 0 \qquad p \in P
\end{aligned}
$$

- we can remove the link between x_{ij} and λ_p variables

Master Problem

- in general, this will have *many more* than 9 variables...

$$
\begin{array}{lrrrrrrrrrll}
\min & 3\lambda_{1246} + & 14\lambda_{12456} + & 5\lambda_{1256} + & 13\lambda_{13246} + & 24\lambda_{132456} + & 15\lambda_{13256} + & 16\lambda_{1346} + & 27\lambda_{13456} + & 24\lambda_{1356} & & \\
\text{s.t.} & 18\lambda_{1246} + & 14\lambda_{12456} + & 15\lambda_{1256} + & 13\lambda_{13246} + & 9\lambda_{132456} + & 10\lambda_{13256} + & 17\lambda_{1346} + & 13\lambda_{13456} + & 8\lambda_{1356} & \leq & 14 \\
& \lambda_{1246} + & \lambda_{12456} + & \lambda_{1256} + & \lambda_{13246} + & \lambda_{132456} + & \lambda_{13256} + & \lambda_{1346} + & \lambda_{13456} + & \lambda_{1356} & = & 1 \\
& \lambda_{1246}\;, & \lambda_{12456}\;, & \lambda_{1256}\;, & \lambda_{13246}\;, & \lambda_{132456}\;, & \lambda_{13256}\;, & \lambda_{1346}\;, & \lambda_{13456}\;, & \lambda_{1356} & \geq & 0
\end{array}
$$

Restricted Master Problem

- in general, this will have *many more* than 9 variables...

$$
\begin{array}{llllr}
\min & 5\lambda_{1256} + 13\lambda_{13246} & + 15\lambda_{13256} & & \\
\text{s.t.} & 15\lambda_{1256} + 13\lambda_{13246} & + 10\lambda_{13256} & \leq & 14 \\
& \lambda_{1256} + \lambda_{13246} & + \lambda_{13256} & = & 1 \\
& \lambda_{1256}\ ,\ \lambda_{13246} & ,\ \lambda_{13256} & \geq & 0
\end{array}
$$

- the restricted master problem works with a (very small) subset of variables only
- we add more variables as needed...

Restricted Master Problem (RMP)

- how do we *generate* such a *column*?

$$
\begin{array}{llllllll}
\bar{z} = \min & \ldots & + & 24\lambda_{132456} & + & \ldots & & \\
\text{s.t.} & \ldots & + & 9\lambda_{132456} & + & \ldots & \leq & 14 \\
 & \ldots & + & 1\lambda_{132456} & + & \ldots & = & 1 \\
 & \ldots & & 1\lambda_{132456} & & \ldots & \geq & 0
\end{array}
$$

Restricted Master Problem (RMP)

- how do we *generate* such a *column*?

$$
\begin{array}{lcccccccl}
 & & & & & & & & \text{duals} \\
 & & & & & & & & \downarrow \\
\bar{z} = \min & \ldots & + & 24\lambda_{132456} & + & \ldots & & & \\
\text{s.t.} & \ldots & + & 9\lambda_{132456} & + & \ldots & \leq & 14 & \pi_1 \\
 & \ldots & + & 1\lambda_{132456} & + & \ldots & = & 1 & \pi_0 \\
 & \ldots & & 1\lambda_{132456} & & \ldots & \geq & 0 &
\end{array}
$$

- for a specific variable, we can compute the reduced cost:

$$
\begin{aligned}
\bar{c}_{132456} &= 24 - (\pi_1, \pi_0)^t \cdot \begin{pmatrix} 9 \\ 1 \end{pmatrix} \\
&= 24 - 9\pi_1 - 1\pi_0
\end{aligned}
$$

Pricing Subproblem

- in general, for this application, the reduced cost of variable λ_p computes as

$$\bar{c}_p = \sum_{(i,j \in A)} c_{ij} x_{pij} - \Big(\sum_{(i,j \in A)} t_{ij} x_{pij} \Big) \pi_1 - \pi_0$$

- and we are interested in the smallest possible:

$\bar{c}^\star = \min \sum_{(i,j) \in A} (c_{ij} - \pi_1 t_{ij}) x_{ij} - \pi_0$

s. t. the x_{ij} encode a *feasible column*

- If $\bar{c}^\star \geq 0$ then there is *no* improving variable;
- otherwise we found a column to add to the RMP

Pricing Subproblem

- in general, for this application, the reduced cost of variable λ_p computes as

$$\bar{c}_p = \sum_{(i,j \in A)} c_{ij} x_{pij} - \left(\sum_{(i,j \in A)} t_{ij} x_{pij} \right) \pi_1 - \pi_0$$

- and we are interested in the smallest possible:

$\bar{c}^\star = \min \sum_{(i,j) \in A} (c_{ij} - \pi_1 t_{ij}) x_{ij} - \pi_0$

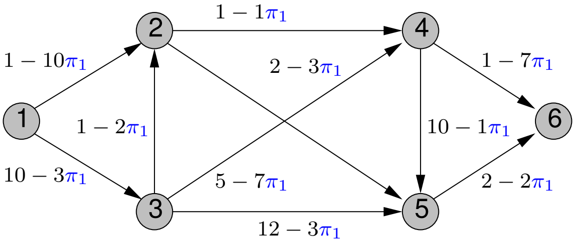
s. t.

$$\begin{aligned}
\sum_{j:(1,j) \in A} x_{1j} &= 1 \\
\sum_{j:(i,j) \in A} x_{ij} - \sum_{j:(j,i) \in A} x_{ji} &= 0 \quad i = 2,3,4,5 \\
\sum_{i:(i,6) \in A} x_{i6} &= 1 \\
x_{ij} &\geq 0 \qquad (i,j) \in A
\end{aligned}$$

- If $\bar{c}^\star \geq 0$ then there is *no* improving variable;
- otherwise we found a column to add to the RMP

Pricing Subproblem

- for this application, this is a shortest path problem in



- this is the original graph with *modified costs*

- remember: this was the reason for the reformulation:
  we wanted to exploit that we can solve shortest path problems

Operations Research

RWTH AACHEN UNIVERSITY

Initializing the Master Problem

- question: how to start?
- initially, we have no feasible solution to the RMP! // maybe no variables at all
- one possibility: "big M approach" // there are several other ways
- introduce artificial variable y_0 with "large" cost, say $M = 100$:

$$
\begin{array}{lrcr}
\bar{z} = \min & 100 y_0 & & \\
\text{s.t.} & & \leq & 14 \\
 & y_0 & = & 1 \\
 & y_0 & \geq & 0
\end{array}
$$

Solving the Master Problem

$$
\begin{array}{llrrl}
\bar{z} = \min & 100y_0 & & & \\
\text{s.t.} & & \leq & 14 & \color{blue}{\pi_1} \\
 & y_0 & = & 1 & \color{magenta}{\pi_0} \\
 & y_0 & \geq & 0 &
\end{array}
$$

master solution	$\bar{z}$	π_0	π_1	$\bar{c}^\star$	p	c_p	t_p

Solving the Master Problem

$$
\begin{array}{llrl}
\bar{z} = \min & 100y_0 & & \\
\text{s.t.} & & \leq 14 & \pi_1 \\
& y_0 & = 1 & \pi_0 \\
& y_0 & \geq 0 &
\end{array}
$$

master solution	$\bar{z}$	π_0	π_1	$\bar{c}^\star$	p	c_p	t_p
$y_0 = 1$	100.0	100.00	0.00				

Solving the Master Problem

$$\begin{array}{lllr}
\bar{z} = \min & 100y_0 & & \\
\text{s.t.} & & \leq & 14 \quad \pi_1 \\
& y_0 & = & 1 \quad \pi_0 \\
& y_0 & \geq & 0
\end{array}$$

master solution	$\bar{z}$	π_0	π_1	$\bar{c}^\star$	p	c_p	t_p
$y_0 = 1$	100.0	100.00	0.00	-97.0	1246	3	18

Solving the Master Problem

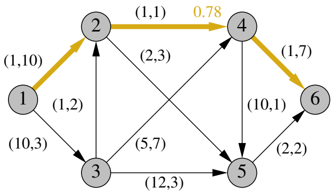
$$
\begin{array}{llcrlr}
\bar{z} = \min & 100y_0 & + & 3\lambda_{1246} & & \\
\text{s.t.} & & & 18\lambda_{1246} & \leq 14 & \pi_1 \\
& y_0 & + & \lambda_{1246} & = 1 & \pi_0 \\
& y_0 & , & \lambda_{1246} & \geq 0 &
\end{array}
$$

master solution	$\bar{z}$	π_0	π_1	$\bar{c}^\star$	p	c_p	t_p
$y_0 = 1$	100.0	100.00	0.00	-97.0	1246	3	18

Solving the Master Problem

$$
\begin{array}{lrcrcrl}
\bar{z} = \min & 100y_0 & + & 3\lambda_{1246} & & & \\
\text{s.t.} & & & 18\lambda_{1246} & \leq & 14 & \color{blue}{\pi_1} \\
 & y_0 & + & \lambda_{1246} & = & 1 & \color{magenta}{\pi_0} \\
 & y_0 & , & \lambda_{1246} & \geq & 0 &
\end{array}
$$

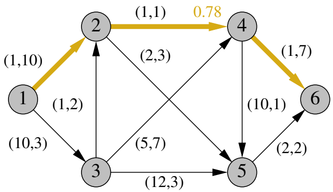
master solution	$\bar{z}$	π_0	π_1	$\bar{c}^\star$	p	c_p	t_p
$y_0 = 1$	100.0	100.00	0.00	$-$ 97.0	1246	3	18
$y_0 = 0.22, \lambda_{1246} = 0.78$	24.6	100.00	-5.39				



Solving the Master Problem

$$
\begin{array}{lrcrcrll}
\bar{z} = \min & 100y_0 & + & 3\lambda_{1246} & + & 24\lambda_{1356} & & \\
\text{s.t.} & & & 18\lambda_{1246} & + & 8\lambda_{1356} & \leq 14 & \color{blue}{\pi_1} \\
 & y_0 & + & \lambda_{1246} & + & \lambda_{1356} & = 1 & \color{magenta}{\pi_0} \\
 & y_0 & , & \lambda_{1246} & , & \lambda_{1356} & \geq 0 &
\end{array}
$$

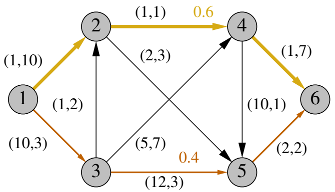
master solution	$\bar{z}$	π_0	π_1	$\bar{c}^\star$	p	c_p	t_p
$y_0 = 1$	100.0	100.00	0.00	-97.0	1246	3	18
$y_0 = 0.22, \lambda_{1246} = 0.78$	24.6	100.00	-5.39	-32.9	1356	24	8



Solving the Master Problem

$$
\begin{array}{lllllll}
\bar{z} = \min & 100y_0 & + & 3\lambda_{1246} & + & 24\lambda_{1356} & & & \\
\text{s.t.} & & & 18\lambda_{1246} & + & 8\lambda_{1356} & \leq & 14 & {\color{blue}\pi_1} \\
 & y_0 & + & \lambda_{1246} & + & \lambda_{1356} & = & 1 & {\color{magenta}\pi_0} \\
 & y_0 & , & \lambda_{1246} & , & \lambda_{1356} & \geq & 0 &
\end{array}
$$

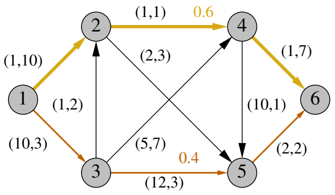
master solution	$\bar{z}$	π_0	π_1	$\bar{c}^\star$	p	c_p	t_p
$y_0 = 1$	100.0	100.00	0.00	$-$ 97.0	1246	3	18
$y_0 = 0.22, \lambda_{1246} = 0.78$	24.6	100.00	-5.39	$-$ 32.9	1356	24	8
$\lambda_{1246} = 0.6, \lambda_{1356} = 0.4$	11.4	40.80	-2.10				



Solving the Master Problem

$$
\begin{array}{lllllllll}
\bar{z} = \min & 100y_0 & + & 3\lambda_{1246} & + & 24\lambda_{1356} & + & 15\lambda_{13256} & & & \\
\text{s.t.} & & & 18\lambda_{1246} & + & 8\lambda_{1356} & + & 10\lambda_{13256} & \leq & 14 & \pi_1 \\
& y_0 & + & \lambda_{1246} & + & \lambda_{1356} & + & \lambda_{13256} & = & 1 & \pi_0 \\
& y_0 & , & \lambda_{1246} & , & \lambda_{1356} & , & \lambda_{13256} & \geq & 0 &
\end{array}
$$

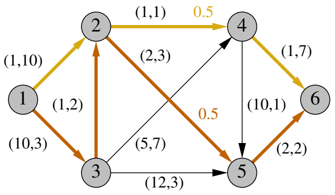
master solution	$\bar{z}$	π_0	π_1	$\bar{c}^\star$	p	c_p	t_p
$y_0 = 1$	100.0	100.00	0.00	$-$ 97.0	1246	3	18
$y_0 = 0.22, \lambda_{1246} = 0.78$	24.6	100.00	-5.39	$-$ 32.9	1356	24	8
$\lambda_{1246} = 0.6, \lambda_{1356} = 0.4$	11.4	40.80	-2.10	$-$ 4.8	13256	15	10



Solving the Master Problem

$$
\begin{array}{llrcrcrcrcl}
\bar{z} = \min & 100y_0 & + & 3\lambda_{1246} & + & 24\lambda_{1356} & + & 15\lambda_{13256} & & & \\
\text{s.t.} & & & 18\lambda_{1246} & + & 8\lambda_{1356} & + & 10\lambda_{13256} & \leq & 14 & \pi_1 \\
 & y_0 & + & \lambda_{1246} & + & \lambda_{1356} & + & \lambda_{13256} & = & 1 & \pi_0 \\
 & y_0 & , & \lambda_{1246} & , & \lambda_{1356} & , & \lambda_{13256} & \geq & 0 &
\end{array}
$$

master solution	$\bar{z}$	π_0	π_1	$\bar{c}^\star$	p	c_p	t_p
$y_0 = 1$	100.0	100.00	0.00	$-$ 97.0	1246	3	18
$y_0 = 0.22, \lambda_{1246} = 0.78$	24.6	100.00	-5.39	$-$ 32.9	1356	24	8
$\lambda_{1246} = 0.6, \lambda_{1356} = 0.4$	11.4	40.80	-2.10	$-$ 4.8	13256	15	10
$\lambda_{1246} = \lambda_{13256} = 0.5$	9.0	30.00	-1.50				

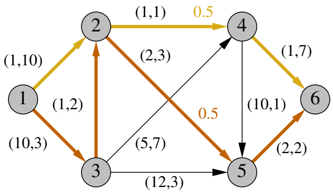


Operations Research | RWTH AACHEN UNIVERSITY

Solving the Master Problem

$$
\begin{array}{llcccccccccccl}
\bar{z} = \min & 100y_0 & + & 3\lambda_{1246} & + & 24\lambda_{1356} & + & 15\lambda_{13256} & + & 5\lambda_{1256} & & & \\
\text{s.t.} & & & 18\lambda_{1246} & + & 8\lambda_{1356} & + & 10\lambda_{13256} & + & 15\lambda_{1256} & \leq & 14 & \pi_1 \\
 & y_0 & + & \lambda_{1246} & + & \lambda_{1356} & + & \lambda_{13256} & + & \lambda_{1256} & = & 1 & \pi_0 \\
 & y_0 & , & \lambda_{1246} & , & \lambda_{1356} & , & \lambda_{13256} & , & \lambda_{1256} & \geq & 0 &
\end{array}
$$

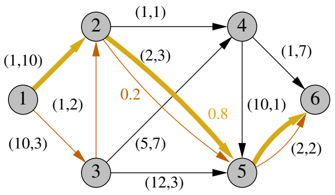
master solution	$\bar{z}$	π_0	π_1	$\bar{c}^\star$	p	c_p	t_p
$y_0 = 1$	100.0	100.00	0.00	$-$ 97.0	1246	3	18
$y_0 = 0.22, \lambda_{1246} = 0.78$	24.6	100.00	-5.39	$-$ 32.9	1356	24	8
$\lambda_{1246} = 0.6, \lambda_{1356} = 0.4$	11.4	40.80	-2.10	$-$ 4.8	13256	15	10
$\lambda_{1246} = \lambda_{13256} = 0.5$	9.0	30.00	-1.50	$-$ 2.5	1256	5	15



Solving the Master Problem

$$
\begin{array}{rrrrrrrrrrrl}
\bar{z} = \min & 100y_0 & + & 3\lambda_{1246} & + & 24\lambda_{1356} & + & 15\lambda_{13256} & + & 5\lambda_{1256} & & \\
\text{s.t.} & & & 18\lambda_{1246} & + & 8\lambda_{1356} & + & 10\lambda_{13256} & + & 15\lambda_{1256} & \leq & 14 \quad \pi_1 \\
& y_0 & + & \lambda_{1246} & + & \lambda_{1356} & + & \lambda_{13256} & + & \lambda_{1256} & = & 1 \quad \pi_0 \\
& y_0 & , & \lambda_{1246} & , & \lambda_{1356} & , & \lambda_{13256} & , & \lambda_{1256} & \geq & 0
\end{array}
$$

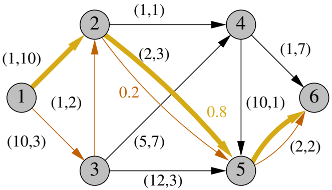
master solution	$\bar{z}$	π_0	π_1	$\bar{c}^\star$	p	c_p	t_p
$y_0 = 1$	100.0	100.00	0.00	-97.0	1246	3	18
$y_0 = 0.22, \lambda_{1246} = 0.78$	24.6	100.00	-5.39	-32.9	1356	24	8
$\lambda_{1246} = 0.6, \lambda_{1356} = 0.4$	11.4	40.80	-2.10	-4.8	13256	15	10
$\lambda_{1246} = \lambda_{13256} = 0.5$	9.0	30.00	-1.50	-2.5	1256	5	15
$\lambda_{13256} = 0.2, \lambda_{1256} = 0.8$	7.0	35.00	-2.00	0			



arc flows: $x_{12} = 0.8,\ x_{13} = x_{32} = 0.2,\ x_{25} = x_{56} = 1$

Solving the Master Problem

- remarks about the optimal RMP solution



- we use 0.2 times path 13256
- we use 0.8 times path 1256, which is *infeasible* in the MP
- a lower bound on the optimal integer solution objective value is $\bar{z} = 7.0$
- we would have obtained the corresponding “arc flow” solution (with the same lower bound) by solving the LP relaxation of the original IP

Do you know it?

- do you know why the dual bound of the reformulation is no better than the LP relaxation of the original formulation?

Do you know it?

- do you know why the dual bound of the reformulation is no better than the LP relaxation of the original formulation?

$\rightarrow$ the polyhedron corresponding to the pricing problem has integer extreme points

$\rightarrow$ solving the pricing problem in integers does not improve over the LP